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Determination of Sphere Sized Distribution from Chord Length Square (CLS) Distribution

Kensuke Baba* and Kiyoshi Miyamoto**

**Department of Pathology and Surgical Pathology, School of Medicine, University of Occupational and Environmental Health, Japan*
***The Laboratory of Medical Sciences, Dokkyo University School of Medicine*

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The capsule model which has been used to extend the Spektor's principle to the case of thick section is used to derive an equation for the determination of the surface-area-proportional sphere size distribution from the distribution of the length square of the test line intercept contained within the sphere. The intercept length square is detected as chord length square (CLS) on the test plate. The range of CLS is divided into k classes of the length square equal interval $\Delta(l^2)$. The number of the i-th class CLS per unit test line length is given by;

$$n_l(i) = \sum_{I \geq i}^{k} \{\frac{\pi}{4}\Delta(l^2) + t\sqrt{\Delta(l^2)}\,\psi(I,i)\}N_v(I),$$

where $n_l(i)$ is number of the i-th class CLS per unit test line length; $N_v(I)$ is number of the I-th sphere per unit test space volume; t is the thickness of the section; $\psi(i,I)$ is $\sqrt{I-(i-1)} - \sqrt{I-i}$; $\Delta(l^2)$ is class interval of CLS. When $t=0$, we obtain an extremely simple relationship;

$$N_v(I) = -\frac{4}{\pi}\cdot\frac{n_l(i+1) - n_l(i)}{\Delta(l^2)}.$$

This equation can be reformed to be a difference equation. It should be empnasized that that does not contain i because that has no conversion matrixes.

INTRODUCTION

Wicksell studied the relationship between the sphere radius distribution (histogram) $F(R)$ and the disk radius distribution (histogram) $f(r)$ on the test plate of extremely thin section ($t=0$) in 1925 ($F(R) \leftarrow f(r,t=0)$). In 1967, Bach extended Wicksell's method to thick section ($F(R) \leftarrow f(r,t)$). Also, Spektor has developed a relationship between the distribution of sphere radii and the length distribution of test line intercepts (l) contained within the sphere for the thin section in 1950 ($F(R) \leftarrow f(l,t=0)$). This finding was extended by Baba, Miyamoto *et al* (1980, 1983) who measured sphere size distribution from chord length distribution using thick section in 1980 ($F(R) \leftarrow f(l,t)$). A capsule like model was used to derive the formula.

The main part of the conversion matrix for ($F(R) \leftarrow f(r,t)$) and the term for thickness in the conversion matrix for $F(R) \leftarrow f(l,t)$ are essentially identical. Their conversion matrix is composed of 120 different values for the 15 grade classification of the size. In this paper, the authors derived a formula for

* Kitakyushu, Fukuoka 807 Japan
** 880 Kitakobayshi, Mibu, Tochigi 321-01 Japan

determination of sphere size distribution from the chord length square (CLS) distribution considering the thickness of section $(F(4\pi R^2)\leftarrow f(l^2,t))$, using the capsule model. Although the term for thickness is expressed by 120 coefficients, they are made of 15 different values for the 15 grade size classifications. Not only the table of coefficients for the thickness term is simple as mentioned above, but also this formula becomes extremely simple as a difference equation when the thickness is zero.

SPHERE AND INTERCEPT SIZING

Assume a lot of different size spherical granules are randomly distributed in a limited 3-D real space of volume v. Suppose test lines of the total length λ are randomly but parallelly drawn in the test space to sense their intercepts contained within the sphere. The surface area of the sphere ranges between 0 and S_{max}. The sphere area range is divided into k classes of area-equal interval $S_{max}/k, (k>1)$.

The test line intercepts contained within the sphere will yield chords intercepted by the disk periphery on the test plate. The intercept is denoted by l. Let us call l^2 simply CLS (chord length square). The CLS ranges between 0 and S_{max}/π. This range is divided also into k classes of equal areas $\Delta(l^2)=(l^2)_{max}/k=S_{max}/\pi k\ (k>0)$. The range of the i-th class CLS is defined by $\Delta(l^2)(i-1)<(l^2)\leqq\Delta(l^2)i$. The square diameter of the I-th class sphere ranges between $\Delta(l^2)(I-1)$ and $\Delta(l^2)I$. The largest I is equal to k, and is written by I_{max}.

Let denote the number of the I-th-class sphere in the test space by $\hat{N}(I)$ and that number per unit test space volume by $N_v(I)$; the number of the i-th class CLS by $n(i)$; and that number per unit test line length by $n_\lambda(i)$; and the $n(i)$ within the contribution from the I-th class sphere by $n(i;I)$.

THICK SECTION

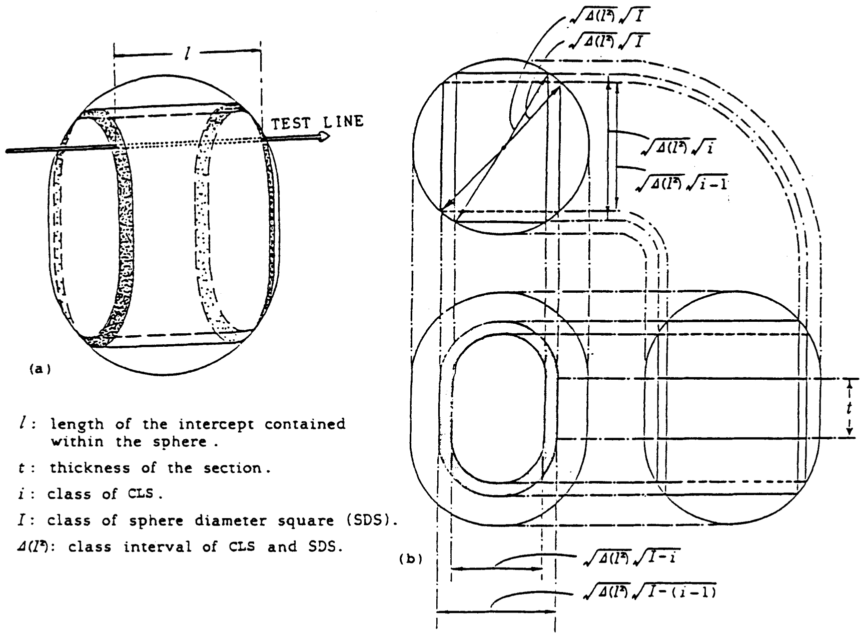
Baba & Miyamoto (1980, 1983) established the capsule model to estimate the probability that a test line is included in a size-defined sphere within the certain intercept for thick section. This model has previously been applied to the unfolding (hitogram-to-hitogram) conversion $F(R)\leftarrow f(l,t)$. Since the capsule model is basically independent of the class interval definition, this model (see Figure) is employed in present study in which the class interval is defined as being $\Delta(l^2)$.

Since the painted area on Figure (b) multiply by test line density, λ/v, into the number of the i-th class CLS for one sphere I, the number of CLS is given by;

$$\frac{\lambda}{v}\{\frac{\pi}{4}\Delta(l^2)+t\sqrt{\Delta(l^2)}(\sqrt{I-(i-1)}-\sqrt{I-i})\}$$

for one sphere I. For all spheres I, the number of CLS of the i-th class is given by

$$n(i;I)=\frac{\lambda}{v}\{\frac{\pi}{4}\Delta(l^2)+t\sqrt{\Delta(l^2)}(\sqrt{I-(i-1)}-\sqrt{I-i})\}N(I). \quad (1)$$



l: length of the intercept contained within the sphere.

t: thickness of the section.

i: class of CLS.

I: class of sphere diameter square (SDS).

$\Delta(l^2)$: class interval of CLS and SDS.

Figure. Capsule Model

(a) Aeroview of the Capsule Model.
(b) Solid Figure of the Capsule Model for the I-th Class Sphere.
All test line intercept lying on the painted area of (a) and (b) will have the length (l) ranging; $\sqrt{\Delta(l^2)}\sqrt{i-1} < l \leqq \sqrt{\Delta(l^2)}\sqrt{i}$ as shown (b). The painted area of (b) is $\frac{\pi}{4}\Delta(l^2) + t\sqrt{\Delta(l^2)}(\sqrt{I-(i-1)} - \sqrt{I-i})$.

Accordingly, the number of the i-th class CLS per unit test line length is given by summing up the contributions from all spheres ranging between $I \geqq i$ and I_{max};

$$n_\lambda(i) = \sum_{I \geqq i}^{I_{max}} \{\frac{\pi}{4}\Delta(l^2) + t\sqrt{\Delta(l^2)}\,\psi(i,I)\}N_v(I), \qquad (2)$$

where $\psi(i,I)$ is $\sqrt{I-(i-1)} - \sqrt{I-i}$ for $i = 1, 2\ 3, \cdots\cdots, I$; and 1 for $I \leqq i$.

The $\psi(I,i)$ is essentially equal to the matrix coefficient for the desk-area-to-surface-area conversion ($F(4\pi R^2) \leftarrow f(2\pi r^2, t=0)$) as proposed by Johnson (1946) and Kimura, Baba *et al* (1978). The inverse equation of Eq. 2 will be obtained for the particular thickness of the section t. For this calculation, the use of a computer is advisable, however, one can get easily each $N_v(I)$ by manual means with stepwise calculation (sequential subtraction) stating from $I = i = I_{max}$, since $\psi(i,I) = 0$ for $I \leqq i$. The 15 values to construct the matrix 15 by 15 coefficients $\psi(i,I)$ are listed in Table 1.

CLASS DEFINITIONS

The coefficients of the conversion matrix, $\psi(i,I)$, are defined as $\sqrt{I-(i-1)} - \sqrt{I-i}$ in the above discussion. We may newly define the conversion coefficients as;

$$\psi(i+½, I) = \sqrt{I-(i-½)} - \sqrt{I-(i+½)} \quad \text{(for } i < I\text{)},$$
$$= \sqrt{½} \quad \text{(for } i = I\text{)},$$
$$= 0 \quad \text{(for } i > I\text{)},$$

and

$$\psi(i, I-½) = \sqrt{(I-½)-(i-1)} - \sqrt{(I-½)-(i)} \quad \text{(for } i < I\text{)},$$
$$= \sqrt{½} \quad \text{(for } i = I\text{)},$$
$$= 0 \quad \text{(for } i > I\text{)}.$$

Both $\psi(i+½, I)$ and $\psi(i, I-½)$ are same and equal to;

$$\psi^*(i,I) = \sqrt{I-i+½} - \sqrt{I-i-½} \quad \text{(for } i < I\text{)},$$
$$= \sqrt{½} \quad \text{(for } i = I\text{)},$$
$$= 0 \quad \text{(for } i > I\text{)}.$$

According to an experimental study to comparatively test the various unfolding methods of $F(R) \leftarrow f(r)$, Cruz·Orive emphasized that bath conversion coefficients ($\phi(i,I)$) derived by Scheil (1931) and its variant $\phi(i+½, I)$ by Wicksell (1925) do not yield faithful results, but the other variant $\phi(i, I-½)$ yields rather faithful results. His conclusion may be correct for the conversion $F(R) \leftarrow f(r)$. In the case of $F(4\pi R^2) \leftarrow f(l^2, t)$, however, we find that $\psi(i+½, I)$ and $\psi(i, I-½)$ are exactly same mathematically.

At least $\psi(i, I)$ should be smaller than 1, since the I class sphere of which diameter is smaller than $\Delta(l^2)I$ does produce smaller number of the I-th class CLS than

$$\frac{\lambda}{v}\{\frac{\pi}{4}\Delta(l^2) + t\sqrt{\Delta(l^2)}\}$$

which is given by Eq.1 for $i = I$. Then, the next equation may be the most faithful.

$$n_\lambda(i) = \sum_{I \geq i}^{I_{max}} \{\frac{\pi}{4}\Delta(l^2) + t\sqrt{(\Delta l^2)}\, \psi^*(i,I)\} N_v(I), \quad \text{(3)}$$

where $\psi^*(i,I)$ is $\sqrt{I-i+½} - \sqrt{I-i-½}$ for $i = 1, 2, 3, \cdots, (I-1)$; $\sqrt{½}$ for $i = I$; and 0 for $i > I$. The values of $\psi^*(i,I)$s for each $I-i$ are listed in Table 1.

THIN SECTION

When the thickness of the section is zero, both Eq. 2 and Eq. 3 become;

$$n_\lambda(i) = \sum_{I \geq i}^{I_{max}} \frac{\pi}{4}\Delta(l^2) N_v(I), \quad (t = 0).$$

The inverse relation of the above equation is give by;

$$N_v(I) = -\frac{4}{\pi}\cdot\frac{n_l(i+1) - n_l(i)}{\Delta(l^2)} \qquad (i = I). \qquad \text{(4)}$$

Eq. 4 can be expressed by a difference equation;

$$N_v(s) = -\frac{4}{\pi}\cdot\frac{\Delta n_l(s)}{\Delta s} \qquad \text{(4')}$$

where s is l^2. The simplicity of Eq. 4′ is certainly one of the important characteristics of this method, but it is even more remarkable that they contain neither I nor i.

SPECIAL REMARKS

Until A. G. Spektor found a method for determining of the sphere size distribution using the so-called linear analysis in 1950, stereologists had estimated the sphere size distribution using "profile analysis" (analysis by detection of a quantity of the profile such as diameter, perimeter, area *etc.*) guided by Abel's integral equation. The differences of the characteristics between these two analyses have been discussed (Baba et al: 1980, 1983: Miyamoto et al: 1985). However the difference between them in basic characteristics has been less examined.

Table 1 : Coefficient ψ

$I-i$	$\psi(i,I)$	$\psi^*(i,I)$
<0	0	0
0	1.000000	0.707107
1	0.414214	0.517638
2	0.317837	0.356394
3	0.267949	0.289690
4	0.236068	0.250492
5	0.213422	0.223888
6	0.196262	0.204305
7	0.182676	0.189103
8	0.171573	0.176863
9	0.162278	0.166731
10	0.154347	0.158163
11	0.147477	0.150795
12	0.141449	0.144369
13	0.136106	0.138701
14	0.131326	0.133652

†: $\sqrt{-\frac{1}{2}} = 0$ in $\psi^*(i, I)$

$\psi(i,I) = \sqrt{I-(i-1)} - \sqrt{I-i}$ for $0 < i \leq I$,

$= 0$, for $i > I$

$\psi^*(i,I) = \psi(i-\frac{1}{2}, I)$

$= \psi(i, I-\frac{1}{2})$

$= \sqrt{I-i-\frac{1}{2}} - \sqrt{I-i-\frac{1}{2}}$, for $0 < i \leq I-1$,

$= 1/2$, for $i = I$,

$= 0$, for $i > I$.

Table 2 : Difference of Coefficient among Various Methods

x	X	$T_0(i,I)$	$T_1(i,I)$	COMMENTS
"profile analyses":				
r	R	ϕ	δ	Extended Scheil's eq. (Bach)
r^2	R^2	ψ	δ	Extended Kimura's or Johnson's eq.
so-called linear analyses:				
l	R	$2I-1$	ϕ	Exptended Spektor's eq. (Baba)
l^2	R^2	1	ψ	Present study

ψ: see Eq. 2.
ϕ: Scheil's coefficient or similar
δ: Kroneker's delta
i: class of sphere diameter
r: radius of the disk
R: radius of the sphere
x, X, T_0, T_1 : see Eq. 5.
l: length of intercept contained within the sphere.

THICK SECTION

For the following discussion, a general equation is proposed;

$$n(x(i,I)) = \{a_0 T_0(i,I) t^0 + a_1 T_1(i,I) t^1\} N(X(I)) \quad \text{..} (5)$$

where a_0 and a_1 are constants; $T_0(i,I)$ and $T_1(i,I)$ are matrix coefficients; t^0 is 1; t^1 is thickness of the section (t); X is a quantity of the sphere such as R and $4\pi R^2$; I is its class; x is the detection quantity with or without modification such as l, l^2, $2r$ and πr^2; i is class of x; $N(X(I))$ is number of $X(I)$s; and $n(x(i;I))$ is number of $n(x(i))$ produced by all I-th spheres (all $X(I)$ s).

Table 2 clearly shows the difference of the conversion matrix coefficients among various methods for determination of the sphere size distribution by linear and "profile" analyses. The so-called linear analysis for the determination of the sphere size distribution is, generally, fit to express $T_0(i,I)$ but not fit for $T_1(i,I)$. In contrast with the so-called linear analysis, the "profile analysis" fit for $T_1(i,I)$ but clearly not fit for $T_0(i,I)$.

THIN SECTION

$T_0(i,I)$ of "profile analysis" contains always both I and i. $T_0(i,I)$ of $F(R) \leftarrow f(l)$ contains one of them. $T_0(i,I)$ of the method of the present study contains none of them.

The most fundermental $T_0(i,I)$ for $F(R) \leftarrow f(r)$ has been derived by Scheil (1931) as;

$$\phi(i,I) = \sqrt{I^2 - (I-i)^2} - \sqrt{I^2 - i^2}$$

Variants of $T_0(i,I)$ such as $\phi(i, I-\frac{1}{2})$ and $\phi(i+\frac{1}{2}, I)$ have been also proposed by different stereologists. In order to find a formula to yield faithful results, many stereologists have dealt chiefly with the problem of how to define class limit and class parameter of i and I contained in $T_0(i,I)$ for the conversion $F(R) \leftarrow f(r)$ by the "profile analysis". E.R. Weibel states in his text book that none of ϕs yield faithful result but $\phi(i, I-\frac{1}{2})$ is rather faithful. ψ for $T_0(i,I)$ to perform $F(R^2) \leftarrow f(r^2)$ has same problem caused by class definition of I and i. Also for the conversion $F(R) \leftarrow f(l)$ according to the so-called linear analysis, two different formulae* have been proposed, and their reliabilities are comparatively discussed.

Essentially, these problems are caused by the existence of I and/or i in the conversion coefficients. Neither I nor i is included in the $T_0(i,I)$ derived in this study as mentioned above. Therefore, we are free from the problems caused by class definition of x and X, as far as the authors' method is used.

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*: One of them was derived consistently by an unfolding method by Spektor in 1950. The other used a discreet method by Cahn & Fullman in 1956, and then was convertered to an unfolding form suitable for analyzing histograms by Suwa in 1977. The authors believe Spektor's method will yield faithful results in so far as the detected data are given in a form of histogram, since a discreet method is merely an approximation as far as the data is given by form of histogram.

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3-3
C: I just want to draw your attention to the fact that the method(for section thickness = 0) was published in J. Microsc. 1983, 131, 291-310. (H. J. Gundersen)