Science on Form: Proceedings of the First International Symposium for Science on Form,
General Editor: S. Ishizaka, Editors: Y. Kato, R. Takaki, and J. Toriwaki, 413–420.

Recognition of Shape and Transformation: An Invariant-theoretical Foundation

Nobuyuki Otsu

Electrotechnical Laboratory, AIST, MITI, 1-1-4 Umezono, Sakuramura, Niihari-gun, Ibaraki 305, Japan

Keywords: Pattern, Shape, Invariant Transformation, Descriptors

Observation of patterns is subject to geometrical transformations, and features extracted generally contain information on both shape and transformation. Thus a basic problem of recognition is how to extract and separate the two kinds of features: invariant features as shape-descriptors and variant features as transformation-descriptors.

In this paper the problem is investigated from a general standpoint of invariant theory using operator analysis. Necessary and sufficient conditions for the descriptors are given in the forms of linear partially differential equations, and it is shown that shape-descriptors and also transformation-descriptors are obtained as the elementary solutions. The results provide theoretical bases and analytic methods in the field of pattern recognition, image processing, and also in computational morphology.

INTRODUCTION

Feature extraction is a key problem in pattern recognition, image processing (measurement), numerical taxonomy, and also in computational morphology. In practical situation, observation of patterns is subject to geometrical transformations such as translation, dilatation, rotation, etc., and features extracted from patterns generally contain two kinds of information: shape information and transformation information, in a mixed form. On the other hand, recognition of patterns consists of two mutually independent aspects: shape recognition and transformation recognition. In fact, the concept of shape is basically independent of (invariant to) transformation, and vice versa. It may be stated as the problem of shape vs size in the context of morphology. Thus a basic problem of recognition is how to extract and separate the two kinds of features: invariant features as shape-descriptors and variant features as transformation-descriptors.

In pattern recognition researches, there have been two different ways of approach to the problem (mainly to the problem of constructing invariant features). One is the normalization: i.e., preprocessing to reform input patterns to "standard" ones with regard to transformation. After input patterns are normalized, any features extracted are assured to be invariant. The other approach is to extract such features as are designed to be invariant to the transformation group under consideration. The auto-correlation function of a pattern or its Fourier transform, the power spectrum, is a well-known example of invariant feature to translation.

Hu (1962) employed the classical algebraic theory of invariants (e.g. Weil: 1946) as a theoretical tool for constructing moment invariants and applied the results to character recognition. Moment invariants have been extended to 3-dimensional cases (Sadjadi & Hall: 1980) and also been considered in a general frame (Teague: 1980).

Amari (1966) developed a theory of normalization in a linear feature space from a viewpoint that a normalization procedure in a less dimensional feature space is much easier than that in a high dimensional pattern space. He studied the condition of possible normalization in a general linear feature space, namely the condition that the linear feature space admits a given transformation group, and thereby showed the invariant-theoretical meaning of Fourier or moment linear features and also the normalization procedures in practice.

Along Amari's line, in this paper the conditions for features to be shape- or transformation-descriptors are investigated in a general theoretical frame using operator analysis, and analytical methods for constructing such descriptors are developed. More directly speaking, given any transformation group, we shall derive such equations that yield the descriptors as the solutions (parts of the results have been obtained in Otsu: 1973, 1981).

DEFINITIONS AND FORMULATION

Pattern Space: Patterns are generally represented by real bounded functions f($\mathbf{r}$) ($\mathbf{r} \in R^N$) with compact supports within a domain D in a N-dimensional Euclidean space R^N. A pattern space is defined by a set of such functions and denoted by P_N. It is a function space, and some topology is introduced by norm $\| \ \|$. For examples, P_1 stands for time-signals, and P_2 for characters, pictures, or images.

Invariant Transformation: Such a geometrical transformation as preserves the shape of a pattern (e.g., translation, dilatation, rotation, etc.) is called an invariant transformation (IT). It in total forms a Lie group (e.g., Chevalley: 1946) and is represented by an operator $T(\boldsymbol{\lambda})$ in pattern space P_N. Usually $T(\boldsymbol{\lambda})$ is a product of several elementary ones (called hereafter EIT's):

$$T(\boldsymbol{\lambda}) = T(\lambda_1, \ldots, \lambda_K) = T_1(\lambda_1) \cdots T_K(\lambda_K) \qquad (1)$$

Each EIT is expressed by a one-parameter linear continuous operator satisfying the followings:

1) Closeness: $T(\lambda)$: $f \in P_N \Rightarrow T(\lambda)f \in P_N$ ($\lambda \in \Lambda$: range of λ)
2) Linearity: $T(\lambda)(af_1 + bf_2) = aT(\lambda)f_1 + bT(\lambda)f_2$
3) Continuity in norm: $\|T(\lambda)f_i - T(\lambda)f\| \to 0$ as $f_i \to f$
4) Continuity in parameter: $T(\lambda)f \to T(\mu)f$ as $\lambda \to \mu$

and group property with respect to the parameter

5) Additive group (AG): $T(\lambda)T(\mu) = T(\lambda+\mu)$, $T(0)=I$ (unit operator)
5') Multiplicative group (MG): $T(\lambda)T(\mu) = T(\lambda\mu)$, $T(1)=I$

For examples in P_1, translation, $T(\lambda)f(x)=f(x-\lambda)$, is AG, and amplitude change, $T(\mu)f(x)=\mu f(x)$, is MG. It should be noted that the difference between AG and MG is merely due to the way of parameter representation; viz., $T(\mu)$ of MG is converted to $T(\lambda)$ of AG by changing parameter as $\mu=\exp(\lambda)$. Thus, taking this into account, our theoretical discussion will be confined to AG case of parameter (λ).

Feature Extraction: Feature extraction is viewed as a mapping Ψ from pattern space P_N to a feature space (practically a vector space of finite dimensions). The mapping Ψ is nonlinear in general and might be represented by a set of nonlinear continuous functionals $z_j=\Psi_j[\,f\,]$, $j=1,\ldots,M$, where f is a pattern, and z_j is the feature value associated (complex value in general).

Problem Statement: Under those preparations, we can here state our problem in a more precise way. Corresponding to the transformation of pattern by $T(\boldsymbol{\lambda})$, the associated feature values generally change from $\Psi_j[f]$ to $\Psi_j[T(\boldsymbol{\lambda})f]$. Then the condition for the features to be shape-descriptors (or invariant features) is that the feature values do not change under IT $T(\boldsymbol{\lambda})$; namely,

$$z_i = \Psi_i[f] = \Psi_i[T(\boldsymbol{\lambda})f]. \tag{2}$$

On the other hand, the condition for the feature $\Psi_k[f]$ to be transformation $T_k(\lambda_k)$, or simply λ_k -descriptor is that the increment of the feature value just gives the value of parameter λ_k of the EIT, being invariant to the other EIT's; namely,

$$\lambda_k = \Psi_k[T(\boldsymbol{\lambda})f] - \Psi_k[f]. \tag{3}$$

This corresponds to the fact that any transformation is of relative in nature, and the parameter λ of $T(\lambda)$ is described in difference.

Hence, our problem is how to construct such descriptors.

ADMISSIBLE LINEAR FEATURE SPACE

As a step to such a directly intractable nonlinear case, we shall consider the elementary case of linear feature extraction.

Linear Feature Extractor (LFE): In the case of linear feature extraction, the mapping is given by a set of linear continuous functionals:

$$z_j = \langle g_j, f\rangle = \int_D g_j(\mathbf{r})f(\mathbf{r})d\mathbf{r} \quad (j=1,\dots,M), \tag{4}$$

or

$$\mathbf{z} = \langle \mathbf{g}, f\rangle, \quad \mathbf{z} = (z_1,\dots,z_M) \in F^M, \quad \mathbf{g} = (g_1,\dots,g_M) \tag{4'}$$

where g_j is a differentiable complex-valued measuring function which uniquely characterizes the linear functional, i.e. linear feature extractor (LFE), and F^M is a linear feature space spanned by the set of z_j.

Conjugate and Induced Operators: Here, we shall introduce some notions of operators. EIT $T(\lambda)$ on pattern f can relatively be regarded as EIT $T^*(\lambda)$ acting on measuring functions $\mathbf{g}$. $T(\lambda)$ or $T^*(\lambda)$ induces a transformation (change) of the feature values $\mathbf{z}$, which is symbolically represented by $\hat{T}(\lambda)$. Then $T^*(\lambda)$ and $\hat{T}(\lambda)$ are called the <u>conjugate</u> and the <u>induced</u> transformations (operators), respectively.

$$\hat{T}(\lambda)\mathbf{z} = \langle \mathbf{g}, T(\lambda)f\rangle = \langle T^*(\lambda)\mathbf{g}, f\rangle \tag{5}$$

In the limit of the differential $\partial/\partial\lambda$ as $\lambda \to u$ (u is the *unit* element of the parameter group, u=0 for AG, and u=1 for MG), we have

$$\hat{\tau}\mathbf{z} = \langle \mathbf{g}, \tau f\rangle = \langle \tau^*\mathbf{g}, f\rangle. \tag{6}$$

Then the infinitesimal transformation operator τ is called the <u>generator</u>, and τ^* and $\hat{\tau}$ are called the conjugate and the induced generators, respectively.

Representation Theorem of EIT: According to the theory of Lie group, a finite transformation $T(\lambda)$ can be represented by its generator τ as follows (It is also valid for $T^*(\lambda)$).

$$T(\lambda) = \exp(\lambda\tau) \tag{7}$$

For example, in the case of translation in P_1, $T(\lambda)f(x)=f(x+\lambda)$, the generator τ turns out to be a differential operator:

$$\tau f(x) = \lim_{\lambda\to u}(\partial/\partial\lambda)T(\lambda)f(x) = \lim_{\lambda\to u}(\partial/\partial\lambda)f(x+\lambda) = (\partial/\partial x)f(x)$$

By applying the representation theorem we have

$$f(x+\lambda) = T(\lambda)f(x) = \exp(\lambda\tau)f(x) = \sum_{k=0}^{\infty}(\lambda^k/k!)(\partial/\partial x)^k f(x).$$

This means that the well-known Taylor expansion is a good example of the representation of translation.

Condition for Recognition: For the recognition of shape and also transformation, it is necessary for F^M to admit the transformation; namely, F^M should be close with respect to $T(\lambda)$ (or transformed **z** should remain in F^M), and the induced transformation $\hat{T}(\lambda)$ should be expressed by parameter λ only, independently of pattern f. Then $\hat{T}(\lambda)$ is reduced to a linear representation of EIT and represented by a matrix $H(\lambda)$ called a multiplier (then so is $T^*(\lambda)$).

$$H(\lambda)\mathbf{z} = \langle H(\lambda)\mathbf{g},\ f\rangle \tag{8}$$

In the same limit as in Eq. 6, we have

$$\hat{\tau} = \tau^* = C \tag{9}$$

where $C=H'(u)$ (differential of $H(\lambda)$ at $\lambda=u$) is a constant matrix called a weight matrix. Conversely, by the representation theorem, we confirm

$$\hat{T}(\lambda) = \exp(\lambda\hat{\tau}) = \exp(\lambda C) = H(\lambda).$$

Hence we have the following theorem.

Theorem 1: A necessary and sufficient condition for F^M (or LFE **g**) to admit EIT $T(\lambda)$ is given by the following equation of **g** (usually a linear partially differential equation system):

$$\tau^*\mathbf{g} = C\mathbf{g} \quad \text{(C: any constant matrix)} \tag{10}$$

Then the locus and the tangent of **z** in the linear feature space F^M are given respectively by

$$\hat{T}(\lambda)\mathbf{z} = H(\lambda)\mathbf{z} = \exp(\lambda C)\mathbf{z} \quad \text{and} \quad \hat{\tau}\mathbf{z} = C\mathbf{z}. \tag{11}$$

Corollary 1: Especially when C is diagonal, LFE $z_j=\langle g_j, f\rangle$ are called relative invariant. Then the Eq. 10 reduces to the eigen-equation of τ^*

$$\tau^* g_j = c_j g_j \quad (c_j\text{: eigenvalue}), \tag{10a}$$

and the measuring function g_j are given by the eigen-functions.

Corollary 2: Further especially when $c_j=0$ (zero eigenvalue), LFE is called absolute invariant. Then

$$\hat{T}(\lambda)z_j = z_j. \tag{11a}$$

Theorem 2: Absolute invariant LFE's ($c_j=0$) are linear shape-descriptors.

Theorem 3: Relative invariant LFE's of weight c_j ($c_j\neq 0$) give transformation-descriptors in the following form:

$$\lambda = [\log\hat{T}(\lambda)z_j - \log z_j]/c_j \tag{12}$$

Absolute and relative invariant linear feature extractors for various transformation groups have intensively been investigated in Otsu (1973).

CONSTRUCTION OF NONLINEAR DESCRIPTORS

Consider nonlinear quantities (analytic functions) of **z** which will indicate shape or transformation. Namely, we shall consider to construct nonlinear descriptors from the admissible linear feature space (ALFS); i.e., shape-descriptors (invariant features) $s_i=\Psi_i(\mathbf{z})$ and transformation-descriptors (variant features) $t_k=\Phi_k(\mathbf{z})$, or in vector forms $\mathbf{s}=\Psi(\mathbf{z})$ and $\mathbf{t}=\Phi(\mathbf{z})$, where

1) dim. **t** = K (number of parameters of IT)
2) dim. **s** + dim. **t** = M (= dim. F^M = dim. **z**)

3) shape dim. = pattern dim. - K (IT dim.)
4) dim. **s** (=M-K) $\leqq$ shape dim.

A couple of **s** and **t** is functionally and informationally equivalent to z and is a sort of decomposition of z into shape-descriptors s and transformation-descriptors **t**. It is noted that since the dimensionality of patterns is very high in general, the essential dimensionality of shape is usually much larger than the finite dimensionality (or number) of shape-descriptors. Therefore, contrary to transformation-descriptors, shape-descriptors describe shape approximately in general cases. Of course, so far as we are concerned with special cases of finite dimensional patterns, for example N-polygonal binary images in P_2, the essential dimensionality of pattern and of shape is finite (2N and 2N-K, respectively), therefore then a set of finite number of functionally independent shape-descriptors (2N-K in number) will be sufficient.

Condition for shape-descriptors: By definition in Eq. 2, the condition for a quantity $\Psi_i(\mathbf{z})$ to be a shape-descriptor (invariant feature) is given by

$$\Psi_i(\hat{T}(\lambda_j)\mathbf{z}) = \Psi_i(\mathbf{z}) = s_i \quad (j=1,\ldots,K)$$

Hence, differentiating this by λ_j yields the following result.
Theorem 4: A necessary and sufficient condition for $\Psi_i(\mathbf{z})$ to be a shape-descriptor is that $\Psi_i(\mathbf{z})$ is a solution of the following linear partially differential equation (LPDE), where symbol · denotes the inner product:

$$\mathrm{grad}\Psi_i(\mathbf{z}) \cdot \hat{\tau}_j\mathbf{z} = 0 \quad (j=1,\ldots,K) \tag{13}$$

Condition for transformation-descriptors: Similarly, the condition for a quantity $\Phi_k(\mathbf{z})$ to be a transformation (λ_k)-descriptor is given by

$$\Phi_k(\hat{T}(\lambda_j)\mathbf{z})-\Phi_k(\mathbf{z}) = \delta_{kj}\lambda_k \quad (j=1,\ldots,K)$$

where δ_{kj} is the Kronecker symbol, $\delta_{kj}=1$ for j=k and $\delta_{kj}=0$ for $j\neq k$. By differentiating with respect λ_j we have
Theorem 5: A necessary and sufficient condition for $\Phi_k(\mathbf{z})$ to be a transformation (λ_k)-descriptor is that $\Phi_k(\mathbf{z})$ is a solution of the following LPDE:

$$\mathrm{grad}\Phi_k(\mathbf{z}) \cdot \hat{\tau}_j\mathbf{z} = \delta_{kj} \quad (j=1,\ldots,K) \tag{14}$$

Here, according to the theory of differential equations, both LPDE's can be reduced to the almost same subsidiary equations, ordinal linear differential equation (LDE) systems of the following type:

$$\frac{dz_1}{\hat{\tau}z_1} = \frac{dz_2}{\hat{\tau}z_2} = \cdots = \frac{dz_M}{\hat{\tau}z_M} \quad \left(= \frac{d\lambda}{1} \right) \tag{15}$$

where it should be noticed that $d\lambda$ for AG is changed to $d\mu/\mu$ for MG.
Hence, so far as ALFS (**z**) and the induced generator $\hat{\tau}$ are obtained for given IT T($\boldsymbol{\lambda}$), the descriptors for shape and also for transformations can be derived and constructed as the elementary solutions, or integral constants, w(**z**)=const. of the above equation. It should be remarked that both nonlinear descriptors for shape and also for transformation can be derived from the identical LDE systems. In fact, according to the theory of LPDE, the general solution of Eq. 13 is given by an arbitrary function of the elementary solutions $\omega(\mathbf{z})$'s of Eq. 15 disregarding the last term. This corresponds to the fact that any function of invariants is also invariant. Therefore we may adopt the elementary solutions as elementary invariants (or shape-descriptors). On the other hand, the

transformation-descriptor, $\Phi_k(z)$ for λ_k, is obtained from an elementary solution of Eq. 15 regarding the last term, i.e. by an integral constant in such a form as $\lambda_k=\omega(z)+\text{const}$.

SOME SIMPLE PRACTICAL EXAMPLES

In order to illustrate the preceding theoretical discussion in a more realistic way, we shall practise some simple examples.

Example-1 (Rectangular Wave-Form Recognition in P_1)
Consider a simple case of P_1. Patterns to be recognized are assumed to be rectangular wave-forms, which can be specified by two parameters; width w and height h. We consider only one EIT, translation:

$$T(\lambda)f(x) = f(x-\lambda) \tag{16}$$

In this case the dimensions of patterns and of shape are 3 (w, h, λ) and 2 (w, h), respectively. Therefore we need at least (and at most in this case) 3 features (LFE's), from which two shape-descriptors (corresponding to w and h) and one transformation-descriptor for translation (λ) will be constructed.

A) **Construction from relative invariant LFE**: The generator τ of the translation $T(\lambda)$ and its conjugate generator τ^* are obtained respectively as follows:

$$\tau = -\partial/\partial x \quad \text{and} \quad \tau^* = \partial/\partial x \tag{17}$$

Therefore the necessary and sufficient Eq. 10a for relative invariant LFE is, in this case, given by

$$(\partial/\partial x)g_j(x) = c_j g_j(x). \tag{18}$$

The solution is $g_j(x)=\exp(c_j x)$ without regard to an irrelevant constant coefficient. This means that the Fourier-Laplace transforms

$$z_j = \langle\exp(c_j x), f(x)\rangle \tag{19}$$

are relative invariant LFE's for translation. Then by Theorem 1, we have

$$\hat{T}(\lambda)z_j = \exp(c_j\lambda)z_j \quad \text{and} \quad \hat{\tau}z_j = c_j z_j. \tag{20}$$

As a minimal set of features to constitute ALFS, we shall here choose the three features corresponding to the weights: $c_1=0$, $c_2\neq0$, $c_3\neq0, c_2$. Then the necessary and sufficient Eq. 15 for nonlinear descriptors is given by

$$dz_1/0 = dz_2/(c_2z_2) = dz_3/(c_3z_3) \ (= d\lambda/1) \tag{21}$$

By solving this LDE system, we can derive nonlinear elementary descriptors as shown in the following.

From the first term of Eq. 21 we have $dz_1=0$, or $z_1=\text{const}$. The second term and the third term provide a solution, $c_3\log z_2-c_2\log z_3=\text{const}$. Thus we have two shape-descriptors:

$$s_1 = z_1 = \langle 1, f\rangle \quad \text{and} \quad s_2 = z_2^{c_3}/z_3^{c_2} \tag{22}$$

On the other hand, one transformation (λ)-descriptor is obtained from the solution of the equation; for example, combining the fourth term with the second term, we have $d\lambda=dz_2/(c_2z_2)$, or $\lambda=(\log z_2)/c_2 + \text{const.}$, therefore we have the following nonlinear transformation-descriptor (confirming Theorem 3):

$$\lambda = [\log\hat{T}(\lambda)z_2 - \log z_2]/c_2 \tag{23}$$

B) **Construction from ALFS**: Consider simple LFE, moments:

$$z = (m_0, m_1, m_2), \text{ where } m_j = \langle x^j, f\rangle = \int_D x^j f(x)dx \tag{24}$$

It is easy to show that these moment features admit $T(\lambda)$ (Theorem 1). In fact, the conjugate generator and therefore the induced generator is given by a constant matrix as shown in the following.

$$\hat{\tau} m_j = \langle \tau^* x^j, f\rangle = \langle (\partial/\partial x) x^j, f\rangle = j m_{j-1} \quad (m_j=0 \text{ for } j<0) \tag{25}$$

Therefore, the Eq. 15 for nonlinear descriptors is given by

$$dm_0/0 = dm_1/m_0 = dm_2/2m_1 \ (= d\lambda/1) \tag{26}$$

The first term shows m_0 = const. The second and the third terms yield another integral constant $m_0m_2-m_1^2$ = const. These two elementary solutions just provide two invariant features, shape-descriptors:

$$s_1 = m_0 \quad \text{and} \quad s_2 = m_0m_2 - m_1^2 \tag{27}$$

The parameter w and h which characterize the shape of rectangular wave-forms are functionally equivalent to the shape descriptors s_1 and s_2. Actually the formers can be expressed in terms of the latters as follows.

$$w = 2(3s_2/s_1)^{1/2} \quad \text{and} \quad h = s_1/w \tag{28}$$

On the other hand, from the second and the implicit fourth term we obtain a transformation-descriptor:

$$\lambda = \text{increment of } m_1/m_0 = \hat{T}(\lambda)[m_1/m_0] - m_1/m_0 \tag{29}$$

These results theoretically confirm our intuition that the integral s_1 and the variance s_2/s_1 of pattern are invariant under translation, and the mean m_1/m_0 represents the relative amount of translation (λ).

Example-2 (Similarity-invariant recognition in P_2)

We shall consider patterns $f(x,y)$ in P_2. They may be pictures, characters, or images on a plane, and may be given in gray level, binary, line-drawings, whatever. As IT (invariant transformation), we shall consider similarity transformation consisting of the following operators.

Translation (AG): $T_T(\alpha,\beta)f(x,y) = f(x-\alpha, y-\beta)$

Amplitude (MG): $T_A(\mu)f(x,y) = \mu f(x,y)$

Dilatation (MG): $T_D(\nu)f(x,y) = f(x/\nu, y/\nu)$

Rotation (AG): $T_R(\theta)f(x,y) = f(x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta)$

The dimensionality of pattern is infinity in this case, and so is the shape dimensionality. The dimensionality of IT is four.

As ALFS, we shall consider normalized central moments (the center of gravity is placed at the origin.):

$$\mu_{pq} = \langle x^p y^q, f\rangle / \langle 1, f\rangle \quad (p+q=>2,\ \mu_{00}=1,\ \mu_{10}=\mu_{01}=0) \tag{30}$$

Since μ_{pq} are absolute invariants for T_T and T_A, we have only to consider the rest T_D and T_R. We shall confine our discussion here to the least and simplest ALFS; i.e., $z=(\mu_{20}, \mu_{11}, \mu_{02})$.

Conjugate generators for T_D and T_R are easily obtained as

$$\tau^*_D = x(\partial/\partial x)+y(\partial/\partial y) \quad \text{and} \quad \tau^*_R = x(\partial/\partial y)-y(\partial/\partial x). \tag{31}$$

Therefore induced generators are given by

$$\hat{\tau}_D z = (2z_1, 2z_2, 2z_3) \quad \text{and} \quad \hat{\tau}_R z = (-2z_2, z_1-z_3, 2z_2). \tag{32}$$

Hence the equations for descriptors are

$$\begin{aligned} &D: dz_1/(2z_1) = dz_2/(2z_2) = dz_3/(2z_3) \ (= d\nu/\nu) \ (MG) \\ \text{and} \\ &R: dz_1/(-2z_2) = dz_2/(z_1-z_3) = dz_3/2z_2 \ (= d\theta/1) \ (AG). \end{aligned} \tag{33}$$

By solving this simultaneous LDE system, we have the following results.

One similarity-invariant shape-descriptor (because 3-2=1 in this case):

$$s_1 = L = \frac{\sqrt{4\mu_{11}^2 + (\mu_{20}-\mu_{02})^2}}{\mu_{20} + \mu_{02}} \tag{34}$$

Transformation-descriptors for T_D and T_R:

$$\text{D: } \nu = \sqrt{[\hat{T}(\nu)\mu_{20}]/\mu_{20}}$$

and (35)

$$\text{R: } \theta = \text{increment of } \frac{1}{2}\tan^{-1}\frac{2\mu_{11}}{\mu_{20}-\mu_{02}}$$

The other transformation-descriptors for T_T and T_A can be derived if we apply our method to the other admissible features, for example ordinal moments.

It is interesting to note that the shape-descriptor L in Eq. 34 is just the measure of "lineality" of the pattern shape and closely related to Karhunen-Loève line fitting (Otsu: 1984). In fact it is shown that the following properties hold:

1) Bounded as $0 \leqq L \leqq 1$.
2) The maximum L=1 is attained if and only if pattern f is a straight line.
3) The minimum L=0 is attained if and only if pattern f is uncorrelated ($\mu_{11}=0$) and isotropic ($\mu_{20}=\mu_{02}$).

CONCLUSION

The problem of shape and transformation recognition has been studied from a general standpoint of invariant theory using operator analysis. Necessary and sufficient conditions for shape- and transformation-descriptors have been shown to be given by linear differential equations. Those (nonlinear) descriptors are obtained as the elementary solutions.

The results will provide theoretical bases and analytic methods in pattern recognition, image processing, and also in computational morphology.

REFERENCE

Amari. S. (1966): A normalization theory of pattern signals in feature space. Journal of IECE Japan, 49, 7, pp.1342-1350 (in Japanese).

Chevalley, C. (1946): Theory of Lie Groups. Princeton Univ. Press.

Hu, M., K. (1962): Visual pattern recognition by moment invariants. IRE Trans., IT-8, 2, pp.179 -187.

Otsu, N. (1973): An invariant theory of linear functionals as linear feature extractors. Bull. Electrotechnical Laboratory, 37, 10, pp.893-913.

Otsu, N. (1981): Mathematical Studies on Feature Extraction in Pattern Recognition. Researches of Electrotechnical Laboratory, No. 818, 210 pages (in Japanese).

Otsu, N. (1984): Karhunen-Loève line fitting and a linearity measure. Proc. 7th Int. Conf. on Pattern Recognition, Montreal, pp.486-489.

Sadjadi, F & Hall, E (1980): Three-dimensional moment invariants. TEEE Trans., PAMI-2, 2, pp.127-136.

Teague, M. (1980): Image analysis via the general theory of moments. J. Opt. Soc. Amer., 70, pp.920-930.

Weil, H. (1946): The Classical Group: Their Invariants and Representations. Princeton Univ. Press.