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Two-dimensional Auto-regressive Model for Analysis and Synthesis of Gray-level Textures

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A non-causal type of two-dimensional model, called the two-dimensional auto-regressive (AR) model, is introduced for representation of images field and characterization of textures. In this model, each gray level of a pixel in a image is represented as a linear weighted summation of gray levels of its neighbour pixels on all sides with addition of a white noise.

Basic properties of the model are discussed. An iterative algorithm is proposed to generate sample images from the model with given parameters. And the model identification from a given image is discussed based on maximum likelihood estimation. A class of gray level texture patterns are well characterized and synthesized by using this model. And application of the two-dimensional AR model to textue image segmentation is described.

1. Introduction

In this paper, a two-dimensional stochastic model, called the two-dimensional auto-regressive (AR) model, is introduced for the representation of image fields. And application of the two-dimensional AR model to analysis and synthesis of gray-level textures is described. Here, we deal with images whose characteristics can be presented with some statistical parameters. Such images are called random images. A class of gray-level texture image is a typical example of random images.

During the last decade considerable amount of work has been done on the modeling of random image fields for a basic study of image processings (Jain:1981). In most of the previous work the extension of one-dimensional random signal theories to two-dimensions has been attempted. In this approach, since a two-dimensional image field is considered as the output of a raster-scan process, a causality is introduced in the representation of image fields. It should be noted, however, that such a causality is related only to the scanning process, not to the statistical properties of images (Larimore:1977). In the model proposed, a gray level of a pixel is represented as a linear weighted summation of gray levels of its neighbour pixels on all sides, that is, the model is of non-causal type.

In the following sections, we discuss basic properties of the model, a generation algorithm of sample image from the model, and the model identification problem. Then, texture analysis by the two-dimensional AR model is described.

2. Two-Dimensional Auto-Regressive Model

A discrete image defined on a $I \times J$ rectangular grid is denoted by $\{x_{ij}\}$ $(i=1,2,\cdots,I,\ j=1,2,\cdots,J)$. When each element x_{ij} is a random variable, $\{x_{ij}\}$ is called a discrete random field.

In this paper, we deal with a class of random difference equation,

$$x_{ij} = \sum\sum_{(p,q)\in D} a_{pq}\, x_{i-p,j-q} + n_{ij} \qquad (1)$$

where $\{n_{ij}\}$ is a Gaussian white noise field with zero-mean and the variance σ_n^2, and D is a rectangular region defined as,

$$D = \{(p,q) \mid -M \le p \le M,\ -N \le q \le N,\ (p,q) \ne (0,0)\} \qquad (2)$$

Fig. 1 shows a schematic diagram representation of the equation (1).

The difference equation (1) is an extension of one-dimensional AR model to two dimensions. Thus, it is called the two-dimensional AR model, but this is of non-causal type because D is not restricted into a half-plane (Jain:1981).

The z-transform of the both sides of (1) gives

$$X(z_1,z_2) = \frac{1}{1 - \sum_p\sum_q a_{pq}\, z_1^{\,p}\, z_2^{\,q}}\, N(z_1,z_2) \qquad (3)$$

Therefore, the spectral density is given as

$$f(\omega_1,\omega_2) = \frac{\sigma_n^2}{4\pi^2}\, \frac{1}{\left| 1 - \sum_p\sum_q a_{pq}\, z_1^{p}\, z_2^{q} \right|^2} \qquad (4)$$

where w_1 and w_2 are the angular spatial frequencies normalized with the Nyquist frequency, and $z_1 = e^{-jw_2}$ and $z_2 = e^{-jw_2}$.

Now, we define $IJ \times 1$ lexicographic ordered vectors $\underline{x}$ and $\underline{n}$ from $\{x_{ij}\}$ and $\{n_{ij}\}$, respectively, as

$$\underline{x} = (x_{11}, x_{12}, \cdots, x_{ij}, \cdots, x_{I,J-1}, x_{IJ})'$$
$$\underline{n} = (n_{11}, n_{12}, \cdots, n_{ij}, \cdots, n_{I,J-1}, n_{IJ})' \qquad (5)$$

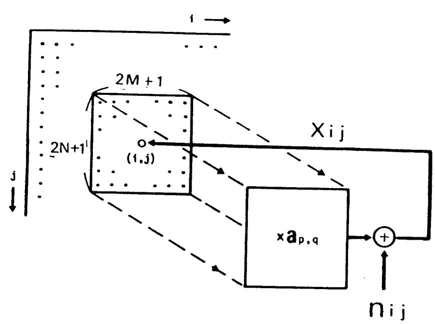

Fig.1. Schematic diagram of the two-dimensional AR model.

Then, the equation (1) is represented as

$$A\underline{x} + \underline{b} = \underline{n} \qquad (6)$$

where A is $(IJ)\times(IJ)$ block Toeplitz type matrix,

$$A = \begin{bmatrix} A_0 & A_{-1} & A_{-2} & \cdots & A_{-M} & & 0 \\ A_1 & A_0 & A_{-1} & & & \ddots & \\ A_2 & A_1 & A_0 & & & & \ddots \\ \vdots & & & \ddots & & & \\ A_M & & & & \ddots & & \\ & \ddots & & & & & \\ 0 & & \ddots & & & & \end{bmatrix} \qquad (7)$$

with

$$\underset{(i\neq 0)}{A_i} = \begin{bmatrix} -a_{i0} & -a_{i,-1} & -a_{i,-2} & \cdots & -a_{i,-N} & & 0 \\ -a_{i1} & -a_{i0} & -a_{i,-1} & & & \ddots & \\ -a_{i2} & -a_{i1} & -a_{i0} & & & & \ddots \\ \vdots & & & \ddots & & & \\ -a_{iN} & & & & \ddots & & \\ & \ddots & & & & & \\ 0 & & \ddots & & & & \end{bmatrix} \qquad (8)$$

where $a_{00}=-1$. The elements of the vector $\underline{b}$ in (6) are to be obtained from the boundary values of the random field defined on the $I\times J$ region. Since we assume that $I,J>>M,N$, and we are interested in the global statistics of $\{x_{ij}\}$, the selection of the boundary values is not important. Thus, in the followings, we put all boundary values to be zero. This gives a simple equation

$$A\underline{x} = \underline{n} \qquad (9)$$

If A is regular, we obtain the covariance matrix X of $\underline{x}$ as

$$X = E[\underline{x}\,\underline{x}'] = E[A^{-1}\underline{n}\,\underline{n}'(A')^{-1}] = \sigma_n^2 (A'A)^{-1} \qquad (10)$$

3. Image Generation by the Two-Dimensional AR Model

In this section we present an iterative algorithm to generate sample images from the two-dimensional AR model with given parameters. A sample image can be obtained as a solution of the linear equation (9) with a sample white noise $\{n_{ij}\}$.

First, we assume that the matrix A is regular. It is also natural to assume that the coefficient $\{a_{pq}\}$ is symmetric, i.e.,

$$a_{pq} = a_{-p,-q} \qquad (11)$$

Then, A is regular and symmetric. In this case, it can be proved that A is positive definite (Deguchi:1980a). Under these conditions, the solution of (9) can be obtained by an iterative algorithm based on the conjugate gradient method (Deguchi et al:1980b). The algorithm is :

i) Initialize the working space $\{ r_{ij} \}$ and $\{ s_{ij} \}$ by setting

$$r_{ij}^{(0)} = s_{ij}^{(0)} = n_{ij} \qquad (12)$$

ii) Iterate 1),2),....5) from n = 0 until the root mean square of $r_{ij}(n)$ becomes smaller than a few percent of that of $x_{ij}(n)$

1) $\alpha^{(n)} = \sum_i \sum_j r_{ij}^{(n)} s_{ij}^{(n)} / \sum_i \sum_j s_{ij}^{(n)} (x_{ij}^{(n)} - \sum_p \sum_q a_{pq} s_{i-p,j-q}^{(n)})$

2) $x_{ij}^{(n+1)} = x_{ij}^{(n)} + \alpha^{(n)} s_{ij}^{(n)}$

3) $r_{ij}^{(n+1)} = n_{ij} - (x_{ij}^{(n+1)} - \sum_p \sum_q a_{pq} x_{i-p,j-q}^{(n+1)}) \qquad (13)$

4) $\beta^{(n)} = \sum_i \sum_j (r_{ij}^{(n+1)})^2 / \sum_i \sum_j (r_{ij}^{(n)})^2$

5) $s_{ij}^{(n+1)} = r_{ij}^{(n)} + \beta^{(n)} s_{ij}^{(n)}$

The result $\{x_{ij}^{(n)}\}$ gives a sample image of $\{x_{ij}\}$.

4. Identification of the Two-Dimensional AR Model

When a sample image $\{ x_{ij} \}$ is given, the two-dimensional AR model can be identified by using the maximum likelihood method. We denote the unknown parameters by $\theta = (\sigma_n^2, \{ a_{pq} \})$.

Since we have assumed that x_{ij} is Gaussian, the logarithm of the conditional probability of the occurrence of an observation $\{ x_{ij} \}$ is given as

$$L(\underline{x}|\theta) = -\frac{IJ}{2}\log 2\pi - \frac{1}{2}\log |X| - \frac{1}{2}\underline{x}' X^{-1} \underline{x} \qquad (14)$$

where $\underline{x}$ is the vector of $\{ x_{ij} \}$ defined in (5), and X is the covariance matrix of $\underline{x}$. It is clear from (10) that X depends only on θ. Maximizing the $L(\underline{x}|\theta)$ with respect to θ yields the maximum likelihood estimate of θ.

When $I,J >> M,N$, we obtain

$$L(\underline{x}|\theta) = -\frac{IJ}{2}[\log 2\pi + \frac{1}{4\pi^2}\iint_0^{2\pi} (\log f(\omega_1,\omega_2) + \frac{S_x(\omega_1,\omega_2)}{f(\omega_1,\omega_2)})d\omega_1 d\omega_2] \qquad (15)$$

where $f(\omega_1,\omega_2)$ is the model spectral density given in (4), and $S_x(\omega_1,\omega_2)$ is the sample spectral density calculated from a sample image $\{ x_{ij} \}$ (Bartlett:1966). An evaluation of the integral using (4) and the Perseval's theorem gives

$$\log \frac{\sigma_n^2}{4\pi^2} + \log k + \frac{1}{\sigma_n^2}\frac{E}{IJ} \qquad (16)$$

where

$$k = \exp(-\frac{1}{4\pi^2} \iint_0^{2\pi} \log |1 - \sum\sum a_{pq} z_1^p z_2^q|^2 d\omega_1 d\omega_2) \qquad (17)$$

and

$$E = \sum_i \sum_j e_{ij}^2 \qquad (18)$$

$$e_{ij} = x_{ij} - \sum_p \sum_q a_{pq} x_{i-p,j-q} \qquad (19)$$

E is the sum of square errors in the model fitting. From the condition that $\partial L(\underline{x}|\theta)/\partial\sigma_n^2 = 0$, we obtain

$$\sigma_n^2 = E / IJ \qquad (20)$$

The substitution of this equation into (16) yields

$$\log E + \log k + \text{const.} \qquad (21)$$

Since $|X| = |\sigma_n^2 (A'A)^{-1}| = \sigma_n^2 |A|^{-2}$, $\log k$ can be rewritten as

$$\log k = \frac{-2}{IJ} \log |A| \qquad (22)$$

which shows that k is not necessarily equal to 1. That is, to determine the model the quantity to be minimized with respect to $\{a_{pq}\}$ is $\log kE$, i.e.,

$$kE \longrightarrow \text{minimum.} \qquad (23)$$

5. Texture Analysis Using the Two-Dimensional AR Model

In this section, a texture analysis method based on the two-dimensional AR model is presented. A class of image texture is considered as a typical example of random images. The objective of texture analysis is to characterize the given texture image and partition the image into texture segments. Here, we present a method for texture segmentation by the two-dimensional AR model identification and merging techniques.

First, the given texture image is divided into small regions as shown in Fig. 2. Then, the two-dimensional AR models are identified for each small regions, and the parameters of the models are obtained.

Next, the regions which have the same characteristics of texture are merged iteratively to make single texture area having single texture characteristics. This procedure is shown in Fig. 3. The estimated noise variance σ_n^2's for each regions can be considered as likelihood criteria of the model fitting to the regions. Therefore, whether two adjacent regions belong to one texture segment or not can be tested through this value. To do this, another new model is fitted to the joined region of two adjacent small regions, for example regions R1 and R2 in Fig.3. Then, we evaluate

$$F_1 = \frac{(\sigma_n^2)_1}{(\sigma_n^2)_{12}} \quad \text{and} \quad F_2 = \frac{(\sigma_n^2)_2}{(\sigma_n^2)_{12}} \qquad (24)$$

Regions R1 and R2 are determined to be of the same texture region and merged if both F_1 and F_2 are within some level of significance. This merging and region growing procedure is iterated until no more region can be merged.

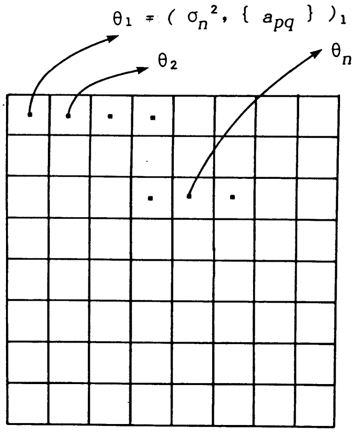

Fig.2. First procedure of texture segmentation. The image is divided into small regions and the AR models are identified for each regions.

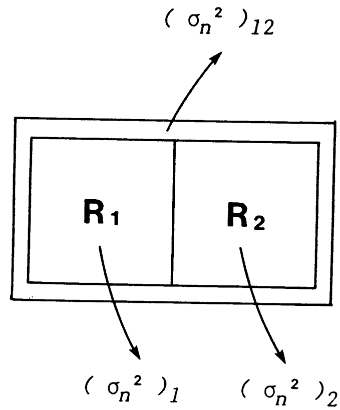

Fig.3. Second merging procedure of texture segmentation. region R1 and R2 are merged when both F1 and F2 of (24) are within some level of significance.

6. Experimental Results

In this section, some experimental results of random texture generation and its parameter identification will be shown.

The two-dimensional AR model used in the experiments is

$$x_{ij} = a (x_{i-1,j} + x_{i+1,j}) + b (x_{i,j-1} + x_{i,j+1}) + n_{ij} \qquad (25)$$

The region D consists of only the four nearest neighbours. Fig. 4 shows four texture images generated using a single white noise $\{ n_{ij} \}$ shown in Fig. 5. The parameter values used are shown in Table 1. The iteration number in the algorithm (12) and (13) is 15 for all images. The image (b) of Fig.4 is smoother, i.e., higher correlated than the image (a). The image (c) has higher correlations in the horizontal direction, on the other hand, the image (d) has higher correlation in the vertical direction. These characteristics meet the given parameters in Table 1, and also the characteristics of the white noise image of Fig.5 hold in all images.

The image of Fig.4(a) has been used for the identification of the two-dimensional AR model (25), assuming $a = b$. Table 2 shows the calculated values of E, k, and kE for various values of a. This table shows that the kE has the minimum value when $a = 0.2$, which agrees with the parameter value used in the image generation.

Finally, the application to texture analysis of an aerial image is shown. Fig. 6(a) is a forest image taken by an airplane. This image has 512x512 pixels. From this image, the characteristics of textures of forest patterns were identified and the image field was segmented into the regions having different parameters. We started the proposed identification and merging procedure with dividing this image into 32x32 pixel small regions. The final result of texture segmentation is shown in Fig.6(b). Texture types have been classified. We can get more fine boundary by further pixel-by-pixel tests starting with this image.

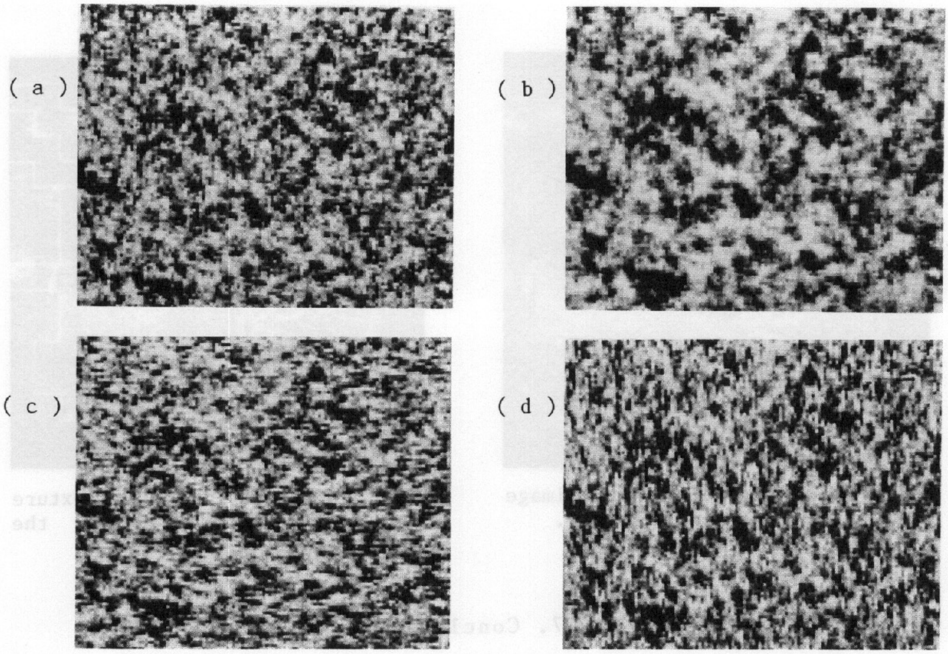

Fig.4. Example of images generated by the two-dimensional AR model of (25). The coefficients are given in Table 1.

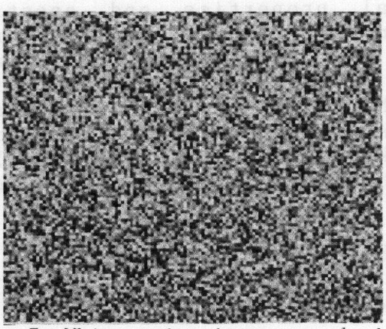

Fig.5. White noise image used in the generation of images of Fig.4.

Table 1 Parameter values used in the generation of images of Fig.4.

Image	a	b
(a)	0.2	0.2
(b)	0.23	0.23
(c)	0.3	0.1
(d)	0.1	0.3

Table 2 Calculated values of E, k, kE with respect to the various values of a for the image of Fig.4(a).

a	0.05	0.1	0.15	0.2	0.22	0.25	E_{min}
E	28233	22615	18399	15585	14851	14172	13882
k	1.01	1.04	1.11	1.22	1.30	1.55	
kE	28514	23519	20422	<u>19012</u>	19308	21966	

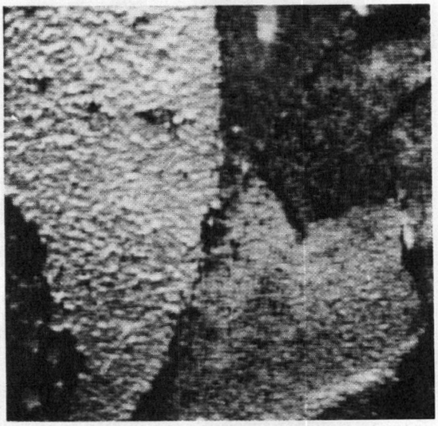

Fig.6. A example of texture image (Aerial photograph of forests).

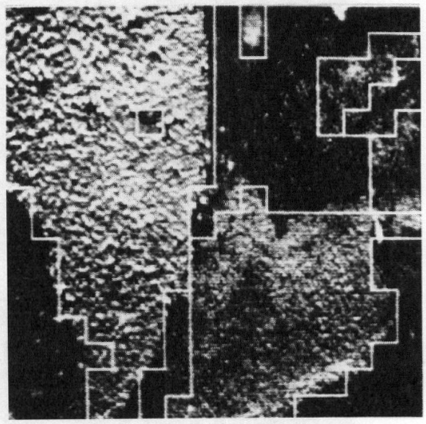

Fig.7. Result of texture segmentation of Fig.6 by the proposed method.

7. Conclusion

A non-causal type of random image field model, called the two-dimensional auto-regressive model, has been introduced for texture characterization. Statistical properties and spectral representation of the model have been discussed. Then the iterative algorithm has been proposed to generate the image represented by the model with given parameters. This generation method can be utilized to generate the random texture images having known statistical properties. And, the model identification from a given image by the maximum likelihood method has been also discussed. The application to the texture analysis has been described.

REFERENCES

Bartlett, M. (1966): An Introduction to Stochastic Process. Cambridge University Press.

Deguchi, K. (1980a): A Generation Model of Two-dimensional Random Fields and Its Applications to Image Processings. Doctoral Dissertation, University of Tokyo.

Deguchi, K. and Morishita, I. (1980b): Image Data Compression and Reconstruction by Two-Dimensional Interpolative Linear Estimation. J.Fac.Engineering, University of Tokyo(B), 35: 537-551.

Jain, A.K. (1981): Advances in Mathematical Models for Image Processing. Proc.IEEE, 69: 502-528.

Larimore, W. (1977): Statistical Inference on Stationary Random Fields. Proc.IEEE, 65: 961-970.

Woods, J.W. (1976): Two-Dimensional Markov Spectral Estimation. IEEE Trans., IT-22: 552-559.

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Q: My question concerns the scope of applicability of your interesting method. I can understand, that, when applying it to aerial images of forests, you will be able to segment the scene according to texture properties. But let's assume, we have a 3D-scene of spherical bodies with textured surfaces; the scene will be lighted from a point source.
Question 1: Will your method extract the same texture properties for all the regions of the textured 3D-surface?
(concern:texture gradient) Did you do experiments?
Question 2: If texture gradient affects the applicability of your method, is it possible to modify it accordingly?
Question 3: Will your method extract the same texture properties for areas of a textured surface, that are shaded? (I assume: yes) (M. Hild)

A: Our texture parameter represents global statistical properties of the texture. If the texture in the objective region is not homogeneous or has some gradient, the extracted parameter represents some averaged properties in the region. So, the availability and the results of our method for the texture having gradients depends on the size of the object region, texture coarseness and smoothness of the gradients. By our method, two groups of parameters are extracted. One group, $[a_{pq}]$, does not depend on the dynamic range of the gray-level of the texture, i.e., darkness or shaded, and represents shape properties of the texture. Only σ_n depends on the darkness of the texture.

Q: Could you comment on the relation of texture parameters to human texture recognition? (E. Hall)

A: It is very difficult, we think, to parameterize the human texture recognitions. Unfortunately, we have no systematic approach to analyze it. But, here we present a method to synthesize texture patterns for such analysis. We will be happy if it will become clear that some relationships hold between human texture recognition and our parameters.